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Improved computation of the adaptation coefficient in the CIE system of mesopic photometry

CHENG GAO,[1] YANG XU,[1] ZHIFENG WANG,[1] MANUEL MELGOSA,[2] MICHAEL POINTER,[3] MING RONNIER LUO,[3,4] KAIDA XIAO, [3] AND CHANGJUN LI[1,5,*]

[1]*School of Electronics and Information Engineering, University of Science and Technology Liaoning, Anshan 114051, China*
[2]*Department of Optics, Faculty of Sciences, University of Granada, Granada 18071, Spain*
[3]*School of Design, University of Leeds, Leeds LS2 9JT, UK*
[4]*State Key Laboratory of Modern Optical Instrumentation, University of Zhejiang, Hangzhou 310058, China*
[5]*College of Physics and Information Engineering, Wenzhou University, Wenzhou 325035, China*
**cjliustl@sina.com*

Abstract: New values of parameters *a* and *b* are proposed for the CIE system of mesopic photometry MES2 [CIE Publication 191:2010], because from the original values this model may have no solution or multi-solutions. From the new values of parameters *a* and *b* it is shown that the CIE MES2 system has a unique solution. The difference however, between the original and the new values of parameters *a* and *b* is very small and the changes do not affect previous conclusions based on the MES2 model. To compute such a solution, we propose a Bisection-Newton method which exhibits fast convergence (8 iterations in the worst case), and improves the fixed-point method recommended by the CIE MES2 system, which has convergence problems for high values of the photopic luminance and very high values of the scotopic/photopic ratio. Comparative results for the fixed-point method, the Bisection method, the Newton method, and the Bisection-Newton method, in terms of the number of iterations necessary for convergence and the computation time used, are reported.

OCIS codes: (120.5240) Photometry; (330.7320) Vision adaptation

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1. Introduction

Mesopic vision is defined as vision intermediate between photopic and scotopic vision [1]. In mesopic vision, both the cones and rods are active; photopic vision, on the other hand, is dominated by cone activity and, in scotopic vision only the rods are active. Mesopic lighting applications are those for which our visual system is operating in a mesopic state, i.e. where both rods and cones contribute to visual functions. However, it is not straightforward to determine whether and how this condition is satisfied in any given practical situation [2]. For example, one important application of mesopic vision is road and street lighting for drivers, motorcyclists, cyclists and pedestrians, since the visual environment in night-time traffic conditions falls largely in the mesopic region.

The current CIE system for mesopic photometry, the CIE MES2 system [3], was first published in 2010. It is in fact an intermediate solution between the USP system published by Rea et al. [4] in 2004 and the MOVE system published by Goodman et al. [5] in 2007. The CIE MES2 system defines the spectral luminous efficiency function for mesopic photometry, $V_{mes}(\lambda)$, in the range from $0.005\ cd\ \mathrm{m}^{-2}$ to $5.0\ cd\ \mathrm{m}^{-2}$, as a convex linear combination:

$$M(m)V_{mes}(\lambda) = mV(\lambda) + (1-m)V'(\lambda) \qquad \text{for } 0 \le m \le 1 \tag{1}$$

where $M(m)$ is a normalizing constant such that $V_{mes}(\lambda)$ attains a maximum value of 1, $V(\lambda)$ and $V'(\lambda)$ are the CIE spectral luminous efficiency functions [6] for photopic and scotopic vision, respectively, and m is the so-called coefficient of adaptation, which value depends on the visual adaptation conditions, in such a way that when $m=0$, $V_{mes}(\lambda)=V'(\lambda)$, and when $m=1$, $V_{mes}(\lambda)=V(\lambda)$. For a given light source with a spectral radiance $E(\lambda)$ (in $Wm^{-2}sr^{-1}nm^{-1}$), assuming 380-780 nm as the visible range, the mesopic luminance, L_{mes}, is given by [3, 6]

$$L_{mes} = \frac{683}{V_{mes}(\lambda_0)} \int_{380}^{780} V_{mes}(\lambda) E(\lambda) d\lambda \tag{2}$$

where $\lambda_0 = 555\ nm$, and the photopic and scotopic luminances, L_p and L_s, are defined as

$$L_p = 683 \int_{380}^{780} V(\lambda) E(\lambda) d\lambda \ , \quad L_s = 1700 \int_{380}^{780} V'(\lambda) E(\lambda) d\lambda \ . \tag{3}$$

Since $V(\lambda_0)=1$, if we define

$$C = V'(\lambda_0) = \frac{683}{1699} \ , \tag{4}$$

then, the mesopic luminance [3] can be obtained from

$$L_{mes} = \frac{mL_p + (1-m)L_s C}{m + (1-m)C} \ . \tag{5}$$

In addition, the CIE MES2 system [3] provides the following relationship between the mesopic luminance, L_{mes}, and the coefficient of adaptation m:

$$m = a + b\ log_{10}(L_{mes}) \ , \tag{6}$$

with

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$$a = 0.7670, \quad b = 0.3334 \quad . \tag{7}$$

From Eq. (5), when $m = 0$, $L_{mes} = L_s$; and when $m = 1$, $L_{mes} = L_p$. Hence, for continuity of the luminance scale from scotopic via mesopic to photopic vision, we should have:

$$\text{If} \quad L_s \le 0.005\, cd\ \text{m}^{-2} \quad \text{then} \quad \text{m} = 0 \quad \text{and} \quad L_{mes} = L_s \tag{8}$$

$$\text{If} \quad L_p \ge 5.0\, cd\ \text{m}^{-2} \quad \text{then} \quad \text{m} = 1 \quad \text{and} \quad L_{mes} = L_p \ , \tag{9}$$

with the mesopic range being fixed as:

$$L_s > 0.005\, cd\ \text{m}^{-2} \text{ and } L_p < 5.0\, cd\ \text{m}^{-2} \ . \tag{10}$$

The CIE MES2 system recommends [3] that the coefficient of adaptation m be determined using a fixed-point iteration method [7] based on Eqs. (5)-(7). Specifically, it is indicated to start from photopic and scotopic luminance values, L_p and L_s, assuming $m_0 = 0.5$, and then repeat the following Eqs. (11) and (12) until 'convergence':

$$L_{mes,n} = \frac{m_n L_p + (1 - m_n) L_s C}{m_n + (1 - m_n) C} \ , \tag{11}$$

$$m_{n+1} = a + b * log_{10}\left(L_{mes,n}\right) \quad . \tag{12}$$

In this algorithm 'convergence' means that when two consecutive values m_n and m_{n+1} are close enough, i.e. when

$$| m_{n+1} - m_n | \le \varepsilon \ , \tag{13}$$

where ε is a very small fixed tolerance, the iteration is stopped, and m_{n+1} is accepted as a solution for m. In computations in the current paper we will use $\varepsilon = 10^{-5}$.

Since $L_s = \left(L_s / L_p\right) L_p$, CIE MES2 provides examples to show how the above fixed-point iteration method works using different inputs for L_p and the ratio $\left(L_s / L_p\right)$, abbreviated as S/P from now on [3]. Specifically, it is reported in [3] that for any fixed value of L_p, L_{mes} increases with an increase of the ratio S/P, which agrees with experimental results [8]. However, we think that some questions related to the CIE MES2 system have not yet been addressed. For example: 1) Do Eqs. (5)-(7) determine a unique solution for m when Eq. (10) is satisfied? 2) Does the fixed-point iteration method in Eqs. (11)-(13) converge when Eqs. (5)–(7) determine a unique solution for m ? The current paper investigates these two questions. From our results, a new Bisection-Newton method is proposed to predict the value of the coefficient of adaptation, m, and the mesopic luminance, L_{mes}. Numerical results show that the Bisection-Newton method always converges, and provides a better solution than the fixed-point iteration method currently recommended by the CIE MES2 system [3].

2. Do Eqs. (5)-(7) determine a unique solution for the adaptation coefficient?

Firstly, it follows from Eq. (6) that

$$L_{mes} = 10^{(m-a)/b} \ . \tag{14}$$

Thus, if we let

$$F(m) = \frac{m L_p + (1 - m) L_s C}{m + (1 - m) C} - 10^{\frac{m-a}{b}} \ , \tag{15}$$

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it is clear that the question becomes: Does the function $F(m)=0$ have a unique solution m between 0 and 1 when Eq. (10) is satisfied? From Eq. (15) we note that

$$F(0)=L_s-10^{-a/b} \quad , F(1)=L_p-10^{(1-a)/b} \ . \tag{16}$$

Thus, when

$$L_s>10^{-a/b} \quad \text{and} \quad L_p<10^{(1-a)/b} \tag{17}$$

we have

$$F(0)=L_s-10^{-\frac{a}{b}}>0 \quad \text{and} \quad F(1)=L_p-10^{\frac{1-a}{b}}<0 \ . \tag{18}$$

Therefore, from Bolzano's theorem [9], $F(m)=0$ has at least one solution when Eq. (18) is satisfied. Note that conditions (10) and (17) are approximately the same, if the computation is carried out by hand, since from Eq. (7)

$$10^{-a/b}\approx 0.005006\approx 0.005 \quad \text{and} \quad 10^{\frac{1-a}{b}}\approx 4.998736\approx 5.0 \ . \tag{19}$$

However, when using a computer, the two conditions (10) and (17) are different and this can have a harmful effect as shown by the examples in the next subsection.

2.1 Examples with no solution or multi solutions for F(m) = 0

First, let's assume $L_p=4.999$, and $L_s=(S/P)\,4.999$. If, for example, $(S/P)=0.5$, from Eq. (18) we have $F(0)>0$ and $F(1)>0$, and therefore we are not certain if $F(m)=0$ has a solution or not. In fact, in this case $F(m)>0$ for all m values between 0 and 1, as shown in Fig. 1, where the cases with $(S/P)=1,2$ and 3 have also been plotted. Thus, in all these cases $F(m)=0$ has no solution, even though condition in Eq. (10) is satisfied.

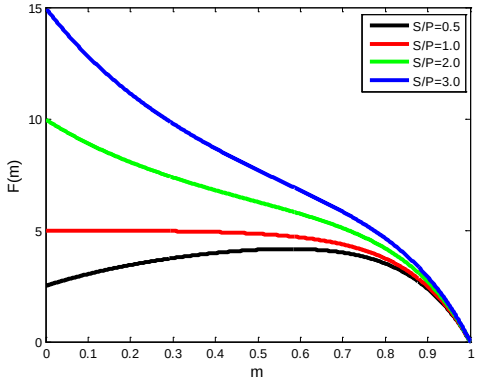

Fig. 1. For $L_p=4.999$ and $L_s=(S/P)\,4.999$, with $S/P=0.5, 1.0, 2.0,$ and 3.0, $F(m)>0$ for $0\le m\le 1$.

Second, let's assume $L_s=0.005004$ and, for example, $L_p=3$. In this case, from Eq. (18), we have $F(0)=-1.65.10^{-6}$ and $F(1)=-1.999$. Because $F(0)<0$ and $F(1)<0$, once again we are not certain if $F(m)=0$ has a solution or not [9]. Since $L_s=0.005004$ is

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very close to 0.005, we expect m to be close to zero, and $F(0) = -1.65 \cdot 10^{-6}$ confirms this is the case. However, as shown by the green curve in Fig. 2, there is another solution with an unexpected high m value (approximately, $m = 0.92116$). In fact, in this case, the fixed-point iteration method recommended by CIE [3] also gives a solution of $m = 0.92116$ and $L_{mes} = 2.9$, which is certainly a wrong solution. Note that in this example Eq. (10) holds. Similar results can be found with $L_s = 0.005004$ and $L_p = 1, 2$ and 4, as shown by the remaining curves plotted in Fig. 2.

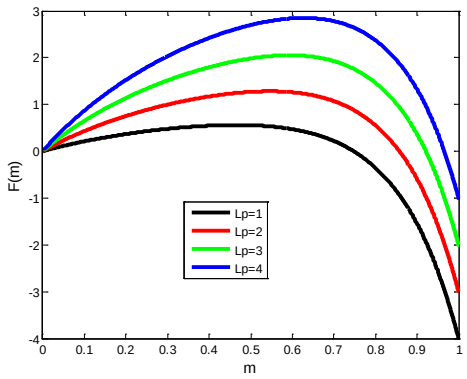

Fig. 2. For $L_s = 0.005004$ and $L_p = 1, 2, 3$ and 4, the function $F(m) = 0$ has wrong solutions with high m values..

2.2 Theorem of existence of an unique solution for F(m) = 0 within the mesopic range

The examples in subsection 2.1 do show that the parameters a and b as defined by CIE in Eq. (7) have problems. In order to redefine these parameters, we note that, if $F(m) = 0$ has a unique solution m between 0 and 1, then when m is small, L_{mes} should be close to 0.005, and when $m = 0$, $L_{mes} = 0.005$. Similarly, when m is high, L_{mes} should be close to 5, and when $m = 1$, $L_{mes} = 5$. Thus, from Eq. (14), the parameters a and b should satisfy

$$0.005 = 10^{-a/b} \quad , \quad 5 = 10^{(1-a)/b} \quad , \tag{20}$$

and by solving Eqs. (20) we have

$$b = \frac{1}{3} \quad , \quad a = 1 - \frac{log_{10}(5)}{3} \quad . \tag{21}$$

Although a and b as defined by Eq. (21) are not much different from a and b as defined by Eq. (7), the examples in subsection 2.1 and the discussion below show that it is better to use the set defined by Eq. (21). Therefore, from now on, we will assume the MES2 model with a and b as defined by Eq. (21). The difference however, between the original and the new values of the parameters a and b is very small and it was found that it does not affect the main conclusions previously published in the literature.

Theorem: When $L_s > 0.005\, cd\ m^{-2}$ and $L_p < 5.0\, cd\ m^{-2}$ [see Eq. (10)], the function $F(m) = 0$ has a unique solution m between 0 and 1.

Proof: Since a and b satisfy Eq. (21), when $L_s > 0.005\, cd\ m^{-2}$ and $L_p < 5.0\, cd\ m^{-2}$, the inequalities defined by Eq. (18) are true (i.e. $F(0) > 0$ and $F(1) < 0$). Hence, since $F(m)$

defined by Eq. (15) is a continuous function of m, Bolzano's theorem [9] indicates that the equation $F(m)=0$ has at least one solution m between 0 and 1.

To prove that there is only one solution for $F(m)=0$, we consider the derivative of $F(m)$ given by

$$\frac{dF}{dm}=\frac{\left[L_p C\left(1-L_s/L_p\right)\right]}{\left[m+(1-m)C\right]^2}-\frac{1}{b}\ln(10)10^{(m-a)/b} \quad . \tag{22}$$

In Eq. (22) the second summand (the right term including the operation minus) is always negative, and, if $L_s/L_p \geq 1$, the numerator of the first summand (left term) is also non-positive, in such a way that we can conclude that $\frac{dF}{dm}<0$ for any m between 0 and 1, and hence $F(m)$ is a strictly decreasing function of m. Thus, if $L_s/L_p \geq 1$, $F(m)=0$ has only one solution (remember that $F(0)>0$ and $F(1)<0$, and $F(m)$ is a continuous function). For the alternative case, $L_s/L_p<1$, we need to consider the second order derivative of $F(m)$, which is given by

$$\frac{d^2F}{dm^2}=-2(1-C)\frac{\left[L_p\cdot C\cdot\left(1-L_s/L_p\right)\right]}{\left[m+(1-m)C\right]^3}-[\frac{1}{b}\cdot\ln(10)]^2\cdot 10^{(m-a)/b} \quad . \tag{23}$$

When $L_s/L_p<1$, the first summand (left term) in Eq. (23) is negative, and, bearing in mind that the second summand (the right term including the operation minus) in Eq. (23) is also always negative, we can state that $\frac{d^2F}{dm^2}<0$, and therefore, $\frac{dF}{dm}$ is a strictly decreasing function of m. Now, if we assume $\frac{dF}{dm}(0)\leq 0$, we must conclude that $\frac{dF}{dm}<0$ for any m between 0 and 1, because $\frac{dF}{dm}$ is a strictly decreasing function of m, which implies that $F(m)=0$ has a unique solution m between 0 and 1 (from the hypothesis of the theorem, $F(0)>0$, $F(1)<0$, and $F(m)$ is a continuous function). Alternatively, if we assume $\frac{dF}{dm}(0)>0$, it must be true that $\frac{dF}{dm}(1)<0$, because otherwise we would conclude that $\frac{dF}{dm}(m)>0$ for any m between 0 and 1 since $\frac{dF}{dm}$ is a strictly decreasing function of m, resulting in that $F(m)=0$ will have no solution for m between 0 and 1, which is in contradiction with the fact that $F(m)=0$ has at least one solution m between 0 and 1 (Bolzano's theorem). Therefore, if $\frac{dF}{dm}(0)>0$, it must be concluded that $\frac{dF}{dm}(1)<0$, and Bolzano's theorem [9] can be used again to state that there exists at least a value m_* between 0 and 1 such that $\frac{dF}{dm}(m_*)=0$. Hence, $\frac{dF}{dm}$ is positive for m between 0 and m_*, and $\frac{dF}{dm}$ is

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negative for m between m_* and 1 since $\frac{dF}{dm}$ is a strictly decreasing function of m. In this situation we have that $F(m)$ strictly increases for m between 0 and m_*, and then strictly decreases for m between m_* and 1. Thus, remembering that from the hypothesis of the theorem $F(m)$ is a continuous function with $F(0)>0$ and $F(1)<0$, we can conclude that $F(m)=0$ has a unique solution for m between m_* and 1, and hence $F(m)=0$ has an unique solution m between 0 and 1 since $F(m)=0$ has no solution for m between 0 and m_*, concluding the proof of the theorem.

3. Improved methods to find the solution for *F*(*m*) = 0

Now the problem is how to find the unique solution for $F(m)=0$ [Eq. (15)], under the constraints $L_s > 0.005\ cd\ \mathrm{m}^{-2}$ and $L_p < 5.0\ cd\ \mathrm{m}^{-2}$ [Eq. (10)]. In CIE Publication 191:2010 [3], a simple fixed-point iteration method was suggested for this purpose. Let

$$g(m) = a + b \cdot log_{10}\left[L_{mes}(m)\right] \quad , \tag{24}$$

where, $L_{mes}(m)$ is defined by Eq. (5). Thus, the fixed-point iteration method [7] is equivalent to the iteration (see Eqs. (11) and (12)):

$$m_{n+1} = g(m_n), \text{ for } n = 0, 1, \ldots \tag{25}$$

If the method converges to a limit m^*, then we have

$$m^* = \lim_{n\to\infty} m_{n+1} = \lim_{n\to\infty} g(m_n) = g\left(\lim_{n\to\infty} m_n\right) = g(m^*). \tag{26}$$

Hence, m^* is a fixed point of the function $g(m)$ and that is why the iteration algorithm in [3] is also called a fixed-point iteration method. It was found that this fixed-point algorithm converges very fast for small values of the ratio S/P. However, for large values of the ratio (e.g., $S/P = 19$, $L_p = 3.7$), this method does not converge after 100 iterations [$\varepsilon = 10^{-5}$, see Eq. (13)]. Though how large the ratio S/P can be is debatable, this example shows that, at least from a theoretical point of view, the fixed-point iteration method may converge very slowly or may not converge at all. To find the root of the function $F(m)$, other numerical methods [7], for example, the Bisection and Newton methods, can be also considered. The Bisection method always converges, but it has a low convergence rate in general. The Newton method, when it converges, has a very fast convergence rate. However, the Newton method may not converge if the initial guess is far away from the true solution. Hence, we have designed a hybrid method, named the Bisection-Newton method, because it is based on the Bisection and the Newton methods, and we propose it for optimal solution of $F(m)=0$. The Bisection-Newton method not only always converges (similar to the Bisection method) to the unique solution for any initial guess between 0 and 1, but also converges very fast (similar to the Newton method). For completeness, the next subsections detail the algorithms corresponding to the Bisection, Newton, and Bisection-Newton methods, respectively [7].

3.1 The Bisection method

Initial step: $k=0,\ \ a_k=0,\ b_k=1,\ \ \varepsilon = 10^{-5}$. Then repeat the following steps until $|m_{k+1} - m_k| \le \varepsilon$:

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1) $m_k = (a_k + b_k)/2$

2) if $|F(m_k)| \le \varepsilon$, stop, m_k can be considered as a solution.

3) if $F(m_k) > 0$, $a_{k+1} = m_k$; $b_{k+1} = b_k$;

4) if $F(m_k) < 0$, $a_{k+1} = a_k$; $b_{k+1} = m_k$;

3.2 The Newton method

Initial step: $k = 0$, $m_0 = 0.5$, $a_k = 0$, $b_k = 1$, $\varepsilon = 10^{-5}$. Then repeat the following step until $|m_{k+1} - m_k| \le \varepsilon$ or $|F(m_k)| \le \varepsilon$:

$$m_{k+1} = m_k - \frac{F(m_k)}{\frac{dF}{dm}(m_k)}$$

3.3 The Bisection-Newton method

Initial step: $k = 0$, $m_0 = 0.5$, $a_k = 0$, $b_k = 1$, $\varepsilon = 10^{-5}$. Then, repeat the next steps 1-2 until $|m_{k+1} - m_k| \le \varepsilon$

Step 1:

1.1) if $|F(m_k)| \le \varepsilon$, stop, m_k can be considered as a solution.

1.2) if $F(m_k) > 0$, $a_{k+1} = m_k$; $b_{k+1} = b_k$;

1.3) if $F(m_k) < 0$, $a_{k+1} = a_k$; $b_{k+1} = m_k$;

Step 2:

2.1) Compute

$$m_t = m_k - \frac{F(m_k)}{\frac{dF}{dm}(m_k)}$$

2.2) if m_t is inside the interval $[a_{k+1}, b_{k+1}]$, $m_{k+1} = m_t$;

2.3) if m_t is not inside the interval $[a_{k+1}, b_{k+1}]$, then

$$m_{k+1} = (a_{k+1} + b_{k+1})/2$$

Firstly, we note that, similar to the Newton method, the Bisection-Newton method only needs to compute the function $F(m_k)$ and its derivative $\frac{dF}{dm}(m_k)$ once per iteration. Secondly, the length of the interval $[a_k, b_k]$ for the Bisection-Newton method decreases as in the Bisection method [7] from 2.2) and 2.3) in Step 2. Thirdly, the above method mainly uses the Newton iteration (see 2.1) and 2.2) in Step 2). However, when the Newton iteration does not generate a good estimation (see 2.2) in Step 2), the method switches to the Bisection method (see 2.3) in Step 2), which is the reason the method is called the Bisection-Newton method. Finally, the Bisection-Newton iteration generates the sequence m_k and the interval

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end point sequences a_k and b_k. As discussed above, we know $F(0)>0$ and $F(1)<0$. Thus, these three sequences satisfy the condition that m_k is always inside the interval $[a_k,\ b_k]$, $F(a_k)>0$ and $F(b_k)<0$.

In addition, the length of the interval $[a_k,\ b_k]$ is decreasing in the Bisection-Newton method. Thus, it is expected that the Bisection-Newton method will converge, which is the merit of Bisection method [7], and when it converges, it will converge quadratically, which is the merit of the Newton method [7]. In summary, the Bisection-Newton method is better than the original Newton method, since the latter may not converge for certain initial guesses, and is also better than the Bisection method, since the Bisection method has a low convergence rate. The performance of the Bisection-Newton method together with the fixed-point method (i.e. the method which has been recommended by CIE in [3]), the Bisection method, and the Newton method will be discussed in the next section. However, we have to note that the Bisection method is simpler than the Newton and the Bisection-Newton methods since it only needs to compute the function $F(m_k)$ once per iteration. The fixed-point iteration method, needs to compute the function $g(m)$ defined by Eqs. (24) and (5) once per iteration.

4. Comparative performance of different computational methods

In CIE Publication 191:2010 [3], performance examples were given in terms of the photopic luminance, L_p, and the ratio of scotopic to photopic luminance, S/P. The combination of L_p and S/P must satisfy Eq. (10), with $L_s=(S/P)L_p$. Here we have selected 25 values for L_p, which were sampled from 0.1 to 4.9, in steps of 0.2. With regard to the range of values for the ratio S/P, in [3], CIE considered values from 0.25 to 2.75 in steps of 0.1. However, Nizamoglu et al. [10] investigated this ratio for nanocrystal hybridized LEDs, and reported that it can achieve values as high as 5.15. Hung et al. [11] considered the theoretical spectral radiance of a light source as a vector with 81 components sampled from 380 nm to 780 nm at 5 nm intervals, and then investigated the maximum luminous efficiency of radiation for a certain level of color rendering index and fixed correlated color temperature. Using a similar theoretical strategy, we considered how large the ratio S/P can be, and we found that it can be greater than 50. So we decided to choose values of the ratio up to 50 as a theoretical limit although we recognize that, for currently available light sources, S/P values higher than 5.15 do not exist. Therefore, we have decided to sample the ratio S/P starting from 0.1 to 2 in steps of 0.05, and then from 3 to 49 in steps of 2. Thus, altogether we selected 63 values for the ratio S/P. Hence, in overall, from all selected values of L_p and S/P, we have a total of 1575 (25 x 63) points. A fixed convergence tolerance $\varepsilon=10^{-5}$ was chosen to compare the performance of our four tested methods (fixed-point, Bisection, Newton, and Bisection-Newton) for these 1575 points. For a particular method, the lower the number of iterations it takes for convergence, the better the method performs. The maximum number of iterations was set as 200; that is, all methods are automatically stopped after 200 iterations, though the convergence rule is not satisfied, and in this case, they are considered as divergent.

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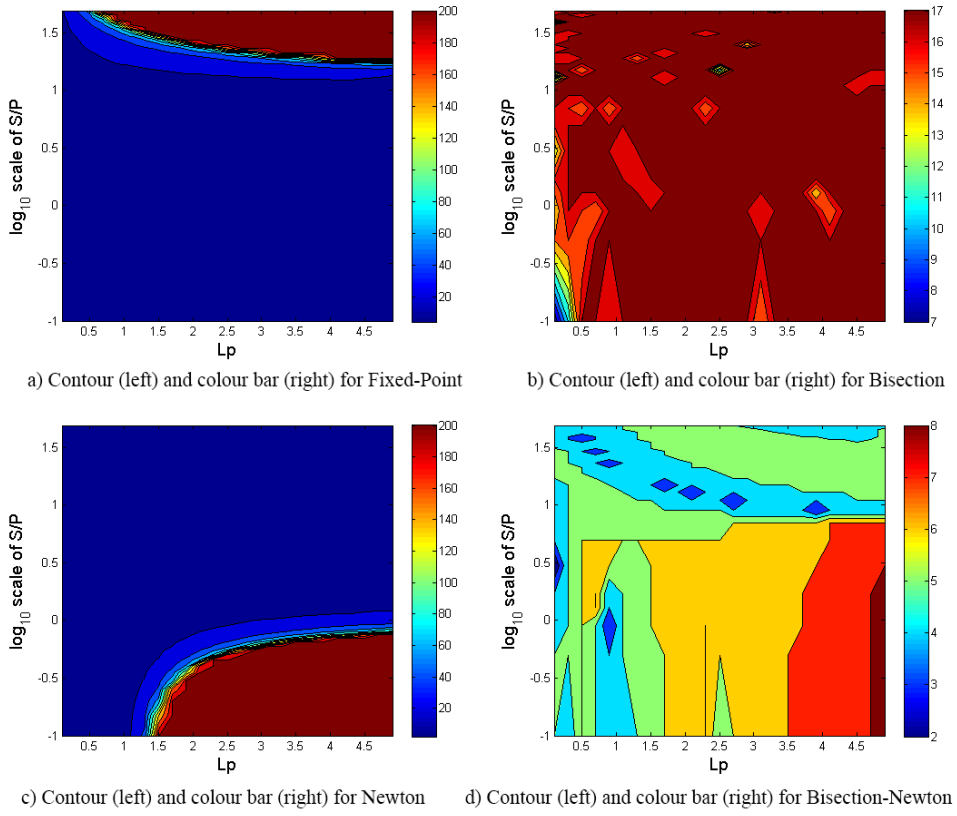

Fig. 3. Contour plots with the number of iterations for convergence as a function of L_p and S/P (in decimal logarithmic scale), considering four computational methods. Color scales for each one of these four plots are different (see vertical bars on the right of each plot)

Figure 3 shows the performances of the four methods in terms of the number of iterations used for each of the 1575 combinations of the photopic luminance L_p and ratio S/P. A decimal logarithmic scale was employed in the charts in Fig. 3 for S/P, to show more detail of the performance for the usual values for current light sources (S/P below 0.75 log units) without discarding potential higher theoretical values. The numbers in the vertical colour bars on the right of each of the four charts in Fig. 3 indicate the number of iterations needed for convergence, and it is important to note that they are different for each one of the four charts. Using the fixed-point iteration method [3] [Fig. 3(a)] it can be seen that for small values of the ratio S/P the method converges fast for all photopic luminances L_p, and also it converges fast for small values of photopic luminance L_p and moderate-high S/P ratios. However, when the ratio S/P and photopic luminance L_p both become large, the number of iterations for convergence in the fixed-point iteration method considerably increases, and for the highest values the method does not converge after 200 iterations. The Bisection method [Fig. 3(b)] always converges, and it takes approximately 17 iterations for all combinations of values of photopic luminance L_p and ratio S/P. In general, the Newton method [Fig. 3(c)] converges very fast, except for very small values of the ratio S/P and moderate-large L_p values, where this method may not converge after 200 iterations. Finally, as expected, we can see that the Bisection-Newton method [Fig. 3(d)] always converges, like the Bisection

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method, and it converges very fast (8 iterations in the worst case), like the Newton method. However, we have to note that the Bisection-Newton methods takes more CPU time per iteration than that used by the Bisection method per iteration (see Table 1 below).

Figure 4 shows another comparison of the relative efficiency (i.e. number of iterations to achieve convergence using a tolerance $\varepsilon = 10^{-5}$) of the four tested methods, assuming a specific value for the photopic luminance, $L_p = 2.1$. Figure 4 plots the number of iterations needed for convergence in each method as a function of the ratio S/P. It can be seen that the fixed-point method (black curve) takes the lowest number of iterations among the four tested methods for values of the S/P ratio smaller than 3. However, when the ratio S/P is greater than 12, it takes the highest number of iterations among the four methods and gradually, it does not converge after 50 iterations when the ratio S/P is greater than 25. Thus, the fixed-point method must not be underestimated because the values of 12 or 25 for the S/P ratio only make sense from a theoretical point of view (currently available light sources have S/P ratios below approximately 5 [10]). For the Bisection method (red curve), about 17 iterations are necessary for convergence for nearly all values of the ratio S/P. The performance of the Newton (green curve) and Bisection-Newton (blue curve) methods is almost identical when the ratio S/P is greater than 5. However, when the ratio S/P is smaller than 5, these two methods perform differently: Specifically, the Newton method took 51 iterations when the ratio S/P was equal to 0.5, and it didn't converge after 200 iterations when the ratio S/P was below 0.35. The proposed Bisection-Newton method (blue curve) took not more than 6 iterations for all values of the ratio S/P.

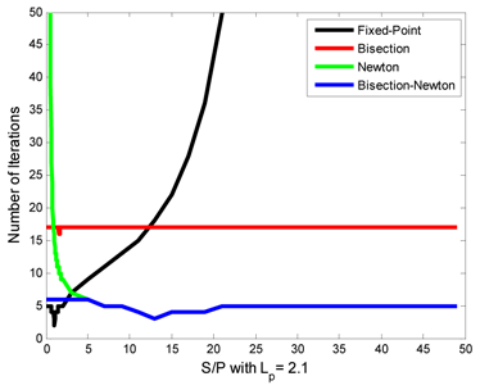

Fig. 4. Number of iterations needed for convergence ($\varepsilon = 10^{-5}$) for each one of the four tested methods as a function of the ratio S/P, assuming a constant photopic luminance, $L_p = 2.1$

Similarly, assuming a ratio $S/P = 1.8$, Fig. 5 plots the number of iterations needed for convergence in each method, as a function of the photopic luminance, L_p. Now, we can see that the four methods converge, the Bisection and fixed-point methods taking the highest and lowest number of iterations for convergence, respectively. The Newton and Bisection-Newton methods perform similarly for L_p values smaller than 0.6, but the Bisection-Newton method is better for larger L_p values. Note that we deliberately selected this small ratio $S/P = 1.8$, which may be typically found in currently available light sources, in order to show that the fixed-point method [3] is still the best for some small S/P ratios.

While Figs. 4 and 5 only illustrate examples that allow for easy comparison of the relative performance of the four methods shown in previous Fig. 3, it can be added that convergence

must be the main goal of any method, and also that the number of iterations considered in Figs. 3-5 is not the only criterion we may consider to evaluate the merit of these methods.

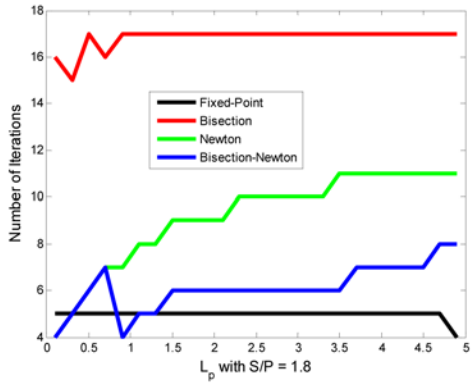

Fig. 5. Number of iterations needed for convergence ($\varepsilon = 10^{-5}$) for each one of the four tested methods as a function of L_p, assuming a constant value for the scotopic/photopic ratio, $S/P = 1.8$

For example, the simplicity of a method, partly related to the time spent in the computation, is also a further criterion which may be considered.

As an example, which may be also useful to readers interested in checking the four methods described in section 3, for each of the these methods Table 1 shows the results found for m and L_{mes}, as well as the number of iterations necessary for convergence with $\varepsilon = 10^{-5}$, and the computational time using a typical desktop computer, for a common light source with $L_p = 2.1; S/P = 1.8$. It can be seen in Table 1 that the values of m and L_{mes} from the four methods are very similar, and the fixed-point method provided the best results from the point of view of the number of iterations and the computation time in seconds (CPU (s)), closely followed by the Bisection-Newton method. This result confirms that, in more practical situations (i.e. low S/P values) we cannot underestimate the fixed-point method currently recommended by CIE [3]. In any case, we can retain our recommendation of the Bisection-Newton method, bearing in mind that it guarantees a fast convergence for a wide range of input values, while the fixed-point method may not converge for high (currently unpractical) values of the S/P ratio. Table 1 also lists, in the last column, the CPU time used per iteration for each method. It can be seen that, for each iteration, the fixed-point method took the most CPU time compared with other methods and the Bisection method took the least. The Bisection-Newton method took twice that used by the Bisection method per iteration.

Table 1. Results found for the four methods described is section 3, assuming $L_p = 2.1; S/P = 1.$

	m	L_{mes}	Number of iterations	CPU (s) used	CPU (s) per iteration
Fixed-point	0.880298	2.187075	5	4.09E–05	0.82 E–05
Bisection	0.880302	2.187072	17	6.74E–05	0.40 E–05
Newton	0.880298	2.187075	9	5.27E–05	0.59 E–05
Bisection-Newton	0.880298	2.187075	6	4.79E–05	0.80 E–05

Furthermore, we note that each of the four methods converges very fast for the example shown in Table 1 but, on repeating the calculations, the CPU time varies. To overcome this we repeated the calculations 1000 times and we quote the average CPU time in Table 1.

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5. Conclusions

We first investigated the parameters a and b for the CIE MES2 model [3]. It was found that from a and b values defined in Eq. (7), the model $F(m)=0$ [Eq. (15)] may have no solution or multi-solutions. Hence new values for parameters a and b were proposed in Eq. (21). Under values in Eq. (21), it was shown that the model $F(m)=0$ has a unique solution m between 0 and 1, for the photopic luminance L_p and the ratio S/P satisfying condition in Eq. (10). Then four methods, including the fixed-point iteration method recommended by the CIE Publication 191:2010 [3], the Bisection method [7], the Newton method [7], and the Bisection-Newton method, originally proposed by us, were used to solve the MES2 model, $F(m)=0$. It was found that the Bisection method converges as long as the value of L_p and the ratio S/P satisfy the condition in Eq. (10) and took approximately the same number of iterations for convergence for any ratio S/P and photopic luminance L_p. For the fixed-point method, it may converge very fast for small values of the ratio S/P, and it is better than all the other methods for ratios S/P smaller than about 3, which is the case for most light sources currently available, but it may diverge for large values of the ratio S/P. The Newton method may diverge for small values of the ratio S/P and it converges for larger values. When it converges, it converges faster than the Bisection method. Finally, the Bisection-Newton method always converges, like the Bisection method, and it converges very fast, like the Newton method.

While convergence can be considered as the most important property of all the test methods, simplicity of the algorithms (e.g. computational time) may be also considered as an added value. Thus, for currently available light sources with an S/P ratio less than 5, the fixed-point iteration method recommended by CIE [3] can still be used. However, for simplicity, the Bisection method can also be used since it is simpler and convergent in any case. In addition, we feel that the Bisection-Newton method should be used for the CIE MES2 model [3] since it converges for all cases, and it is the best method for larger S/P ratios and the second best method for smaller S/P ratios.

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